Complete characterization of the spectrum of the Kitaev model on spin ladders

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We study the Kitaev model on a ladder network and find the complete spectrum of the Hamiltonian in closed form. Closed and manageable forms for all eigenvalues and eigenvectors allow us to calculate the partition function and averages of nonlocal operators in addition to the reduced density matrices of different subsystems at arbitrary temperatures. It is also discussed how these considerations can be generalized to more general lattices, including three-leg ladders and two-dimensional square lattices.

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I. INTRODUCTION

There are very few exactly solvable spin systems in the sense that their complete spectra can be determined in closed form. The most notable one is the Ising model in transverse field or the XY model,[1] where the spectrum can only be determined with the help of highly nonlocal Jordan-Wigner transformation, rendering the calculation of correlation functions very difficult. The other example is the Heisenberg spin chain, where the energy eigenvalues and eigenvectors can only be determined implicitly by solving the system of coupled nonlinear Bethe ansatz equations.[2,3] For a large class of spin models, only the ground state can be found by the matrix product approach.[4–10] Having an exactly solved quantum many-body system, in the sense of complete determination of its spectrum, is always a fortunate situation, which enables one to make a detailed study of thermal and dynamical properties. It also allows one to solve other related systems by perturbation techniques.

While the study of many-body systems has been traditionally done in the community of condensed-matter and statistical physics, and also by mathematical physicists interested in exact solutions, in recent years, these systems have attracted a lot of attention from the quantum information community. The reason is at least twofold: on the one hand, an array or lattice of two-level quantum systems (qubits) is the natural candidate for implementation of quantum information processing tasks and on the other hand, concepts and tools developed in quantum information,[11] which are mainly aimed at characterizing the nature of quantum states such as their entanglements, have been quite useful in understanding different phenomena in such systems, i.e., quantum phase transitions.[12,13]

While in condensed-matter physics, the focus is on the Hamiltonian and interactions, in quantum information, the emphasis is on the quantum states and their properties, i.e., their bipartite and multipartite entanglements. Needless to say, we are not always in the happy situation to have both a physically plausible Hamiltonian on the one hand and an easily obtainable spectrum on the other. For example, in the matrix product formalism, although we can construct ground states with desired symmetries, it is not guaranteed that the parent Hamiltonians are of real experimental interest. Fortunately with the recent advances in optical lattices and cold atoms, we have more freedom in manipulating systems of many-body two-level systems (qubits). Therefore there is less reservation than before in proposing many-body Hamiltonians whose ground states or the low-level excited states may have desirable properties for implementation of quantum information processing.

In this regard, an interesting model has been recently proposed,[14] which has the very desirable feature of showing topological order and anyonic excitations. This was the first model for topological implementation of quantum computation, where qubits are encoded into the homological classes of loops on a surface, which are hence resistant to local errors. The important point is that the topological order is not related to the symmetries of the Hamiltonian and it is robust against arbitrary local perturbations, even those that destroy all the symmetries of the Hamiltonian. We should stress that at present, an implementation of the Kitaev model on a real experimental many-body system may be quite out of reach due to the four-body interactions between the spins; nevertheless, we find it instructive to investigate the properties of this model that at present has mostly theoretical value.

On any lattice, the Kitaev model is defined by the Hamiltonian

$$H = -J\sum_s A_s - K\sum_p B_p, \tag{1}$$

where J and K are positive coupling constants and the vertex operators A_s and plaquette operators B_p are constructed from Pauli operators $X=\sigma_x$ and $Z=\sigma_z$ as follows: A_s is the product of all X's on the links shared by the vertex s and B_p is the product of all Z operators around a plaquette. These operators commute with each other for any geometry of the network.

It is well known that the ground state of this model has symmetries not inherited from the Hamiltonian but from the topology of the surface on which the model is defined. On a genus g surface without boundary, the ground state of model (1) has 4^g-fold degeneracy with a gap, which cannot be removed by local perturbations. Therefore on such a surface, the ground space can encode $2g$ qubits in a way where resistance to errors is automatically ensured by the topology. In recent years, there has been intensive activity on this model and its variations and generalizations in many directions (see the works in Refs. 15–25 and references therein for a sample). The ground states of the Kitaev model can be characterized rather simply in a formal way. Since $A_s^2=B_p^2=I$, the

ground state is the one that is stabilized by all these operators, i.e., $A_s|g\cdot s\rangle = B_p|g\cdot s\rangle = |g\cdot s\rangle$. One such state can be written as $|g\cdot s\rangle = \Sigma_C C_z|\Omega\rangle$, where $|\Omega\rangle$ is a sea of spins in $+x$ direction and $C_z := \Pi_{i\in C} Z_i$ is a product of flipping operators Z around the closed loop C. If C is a homologically trivial loop, i.e., if it is the boundary of a region on the surface, then C_z can be expressed as a product of B_p operators. Since in $|g\cdot s\rangle$ we are summing over all such loops, the action of any B_p on $|g\cdot s\rangle$ leaves it invariant. Since any A_s commutes with all B_p operators and $A_s|\Omega\rangle = |\Omega\rangle$, it is immediate that $A_s|g\cdot s\rangle = |g\cdot s\rangle$ and hence $|g\cdot s\rangle$ is actually a ground state of Eq. (1). The other degenerate ground states are obtained from $|g\cdot s\rangle$ by acting on it by product of flipping operators Z_i around nontrivial homology cycles. (Closed loops that are not boundaries of regions are called homology cycles or just cycles. On genus surfaces these are the same as noncontractible loops, but on higher genus surfaces they are different.)

The excited states are formed by enacting on the ground states by flipping operators along open strings and hence creating two particles,[14,26] called anyons due to their exchange properties. While the ground state of Eq. (1) is rather simple and, in fact, its entanglement properties and that of the related topological color codes[27] have been studied in a number of works,[25,28] a complete characterization of the spectrum is difficult due to the exponentially large number of open string configurations. In other words, it is known that any collection of open strings creates an excited state, but determining the degeneracy of such excited states is not simple due to the above-mentioned difficulty.

One may expect that on a lattice with a simpler structure, these problems can be overcome. In this regard, spin ladders may be of interest not only due to their own interest as systems interpolating between one- and two-dimensional systems but also since they can be used for approximate solution of the more physical two-dimensional systems when the latter can be approximated as an array of ladders with negligible couplings between them.

In fact, the simple structure of ladders facilitates the study of many interesting phenomena that are otherwise difficult to study in general lattices. For example, one of these phenomena is the dynamics of defect production in passing a critical point when a phase transition occurs.[29] This is usually captured in what is known as the Kibble-Zurek scaling law. However it has recently been shown,[30] through a detailed study of a ladder system, i.e., the Creutz ladder,[31] that in systems with topological order, edge states can dramatically modify this scaling law.

It is the purpose of this paper to make a complete study of the Kitaev model on spin ladders. The ladder with periodic boundary condition in one direction has the topology of a cylinder and its ground state is doubly degenerate. We will determine the energy spectrum completely and from there we calculate the partition function and the total entropy. Then we will determine the reduced density matrices and entropy of various subsystems, where it is found how the cylinder topology affects these properties. Finally we discuss briefly how to extend the method to three and higher leg ladders and eventually to the two-dimensional square lattice.

The structure of this paper is as follows. In Sec. II we explain the model and in Secs. II A–II C, we calculate the spectrum, the partition function, and the thermal averages of nonlocal operators. In Sec. III we calculate the density matrices and entropy of various subsystems, which show among other things that there is no entanglement between any two spins at any temperature. We will then show in Sec. V how to extend this method for obtaining the spectrum of the two-dimensional square lattice. The paper ends with a discussion, in which we specially highlight the relation with other works done in the Kitaev model, and an outlook on future research.

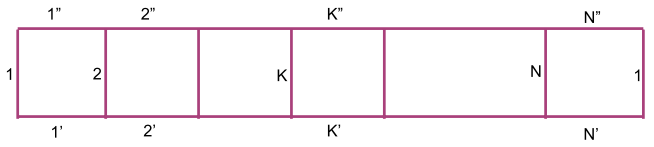

FIG. 1. (Color online) The labeling that will be used in the text for the links on the ladder.

II. KITAEV MODEL ON THE LADDER NETWORK

Consider a two-leg ladder network with length N with the labeling of links as shown in Fig. 1, where we use the superscripts + and − to denote the vertices pertaining to the lower and upper legs. For this ladder, the vertex and plaquette operators take the following form:

$$A_i^+ := X_{i'-1}X_iX_{i'}, \quad A_i^- := X_{i''-1}X_iX_{i''}, \quad B_i = Z_iZ_{i+1}Z_{i'}Z_{i''}. \tag{2}$$

Throughout the paper, we use the generic names A_s and B_p for pointing to general vertex and plaquette operators in a network and the names indicated in Eq. (2) for pointing to specific operators in the ladder. For a closed surface the operators in Eq. (1) are constrained by the relations $\Pi_s A_s = \Pi_p B_p = I$; however, for the ladder, which is a surface with two boundaries, the second constraint does not hold and we are left with

$$\prod_s A_s = I. \tag{3}$$

Therefore the operators in Eq. (2) are $3N-1$ commuting operators in the 2^{3N}-dimensional Hilbert space of the ladder. They all commute with the Hamiltonian and hence the ground state is twofold degenerate. The following nonlocal operators play an important role. Here

$$W_z := \prod_{i'} Z_{i'} \quad W_x := \prod_i X_i. \tag{4}$$

It can be clearly seen that they commute with each other and also with the Hamiltonian,

$$[W_x, W_z] = [H, W_x] = [H, W_z] = 0. \tag{5}$$

They correspond to a cycle around the ladder. Both of them square to 1 and hence have eigenvalues ± 1. As we will see the operator W_z generates the twofold degeneracy of the ground state. Why the operator W_x does not have a similar role? In view of the above commutation relation, one may ask why it does not generate another kind of degeneracy. The

reason will appear in the end of Sec. II B, where we derive the full spectrum.

A. Ground and top states

It is clear from Eq. (1) that the ground state is a state with the property

$$A_s|\Psi_0\rangle = |\Psi_0\rangle, \quad B_i|\Psi_0\rangle = |\Psi_0\rangle, \quad \forall\, s,i. \tag{6}$$

To find the explicit form of the ground state, we use the property $B_i^2 = I$ that is equivalent to $B_i(1+B_i) = 1+B_i$ and introduce the state

$$|\Psi_0\rangle := \frac{1}{\sqrt{2^N}} \prod_i (1+B_i)|\Omega_+\rangle, \tag{7}$$

where $|\Omega_+\rangle := |+\rangle^{\otimes 3N}$ is a product of all spins in the $|+\rangle := |x,+\rangle$ direction. Note that $X|\pm\rangle = \pm|\pm\rangle$ and $Z|\pm\rangle = |\mp\rangle$. It is obvious that $|\Psi_0\rangle$ satisfies condition (6) and hence is a ground state. The other ground state is obtained from $|\Psi_0\rangle$ by the action of the operator W_z, which corresponds to the only nontrivial homology cycle of the surface, that is, $|\Psi_0\rangle$ and $|\Psi_0'\rangle := W_z|\Psi_0\rangle$ form the doubly degenerate ground states of the model,

$$H(|\Psi_0\rangle, |\Psi_0'\rangle) = -N(2J+K)(|\Psi_0\rangle, |\Psi_0'\rangle). \tag{8}$$

Note that W_z, being a cycle, cannot be expressed as any combination of product of plaquette operators B_i, which are all homologically trivial. This shows that the two states $|\Psi_0\rangle$ and $|\Psi_0'\rangle$ are independent.

The top state, the state with the highest energy, is the one that is an eigenstate of all the vertex and plaquette operators with eigenvalues of -1. It can be readily verified that the following state is such a state:

$$|\Psi_{top}\rangle := \frac{1}{\sqrt{2^N}} \prod_i (1-B_i)|\Omega_-\rangle, \tag{9}$$

where $|\Omega_-\rangle = |-\rangle^{\otimes 3N}$ is the product of all spins in the $|-\rangle := |x,-\rangle$ direction in the network. The other degenerate state is obtained by the action of the operator W_z, i.e., $|\Psi_{top}'\rangle = W_z|\Psi_{top}\rangle$. The highest energy will be $E_{top} = N(2J+K)$,

$$H(|\Psi_{top}\rangle, |\Psi_{top}'\rangle) = N(2J+K)(|\Psi_{top}\rangle, |\Psi_{top}'\rangle). \tag{10}$$

B. Complete spectrum

We can now construct the full spectrum. To this end we note that for any arbitrary state $|\chi\rangle$, the state,

$$\prod_i [1+(-1)^{l_i}B_i]|\chi\rangle, \quad l_i = 0,1,$$

is an eigenstate of all the plaquette operators, B_j, with eigenvalues $(-1)^{l_j}$. The reason is the relation $B_j(1\pm B_j) = \pm(1\pm B_j)$. Hereafter we use the abbreviation $B_{\mathbf{l}}$ for such a string of operators,

$$B_{\mathbf{l}} := \prod_i [1+(-1)^{l_i}B_i]. \tag{11}$$

These operators satisfy

$$B_{\mathbf{l}}B_{\mathbf{l}'} = 2^N \delta_{\mathbf{l},\mathbf{l}'} B_{\mathbf{l}}. \tag{12}$$

To construct states that are eigenstates of the vertex operators and at the same time be independent, let us define the following operators:

$$\Lambda_{\mathbf{r},\mathbf{s}} := \prod_i Z_i^{r_i} Z_{i'}^{s_i}, \tag{13}$$

where the sequence of r_i and s_i are 0 or 1 and the labelings are those shown in Fig. 1. Figure 6 shows the links that contribute to the construction of such operators. It is important to note that the links of only one leg are among this set. Now let us define

$$|\Psi_{\mathbf{l},\mathbf{r},\mathbf{s}}\rangle := \Lambda_{\mathbf{r},\mathbf{s}}|\Psi_{\mathbf{l}}\rangle := \frac{1}{\sqrt{2^N}} \Lambda_{\mathbf{r},\mathbf{s}} B_{\mathbf{l}}|\Omega_+\rangle. \tag{14}$$

The operators in Eq. (13) have simple commutation relations with the vertex and plaquette operators. One can verify the following relations:

$$B_j\Lambda_{\mathbf{r},\mathbf{s}} = \Lambda_{\mathbf{r},\mathbf{s}}B_j,$$

$$A_j^-\Lambda_{\mathbf{r},\mathbf{s}} = (-1)^{r_j}\Lambda_{\mathbf{r},\mathbf{s}}A_j^-,$$

$$A_j^+\Lambda_{\mathbf{r},\mathbf{s}} = (-1)^{r_j+s_{j-1}+s_j}\Lambda_{\mathbf{r},\mathbf{s}}A_j^+. \tag{15}$$

This leads to

$$B_j|\Psi_{\mathbf{l},\mathbf{r},\mathbf{s}}\rangle = (-1)^{l_j}|\Psi_{\mathbf{l},\mathbf{r},\mathbf{s}}\rangle,$$

$$A_j^-|\Psi_{\mathbf{l},\mathbf{r},\mathbf{s}}\rangle = (-1)^{r_j}|\Psi_{\mathbf{l},\mathbf{r},\mathbf{s}}\rangle,$$

$$A_j^+|\Psi_{\mathbf{l},\mathbf{r},\mathbf{s}}\rangle = (-1)^{r_j+s_{j-1}+s_j}|\Psi_{\mathbf{l},\mathbf{r},\mathbf{s}}\rangle. \tag{16}$$

The above relations indicate that the states thus constructed are eigenstates of Hamiltonian

$$H|\Psi_{\mathbf{l},\mathbf{r},\mathbf{s}}\rangle = E_{\mathbf{l},\mathbf{r},\mathbf{s}}|\Psi_{\mathbf{l},\mathbf{r},\mathbf{s}}\rangle, \tag{17}$$

where the energy is found from Eqs. (1) and (16) to be

$$E_{\mathbf{l},\mathbf{r},\mathbf{s}}(J,K) = -\sum_j [J[(-1)^{r_j} + (-1)^{r_j+s_{j-1}+s_j}] + K(-1)^{l_j}]. \tag{18}$$

In terms of Ising-like variables, $R_i := (-1)^{r_i}$, $S_i := (-1)^{s_i}$, and $L_i := (-1)^{l_i}$, which take values of ± 1, the energy can be rewritten in the form

$$E_{\mathbf{l},\mathbf{r},\mathbf{s}} = -\sum_j [JR_i(S_iS_{i+1}+1) + KL_i]. \tag{19}$$

In the sequel we will use both the indices r,s,l and R,S,L so that no confusion arises. The states $|\Psi_{\mathbf{l},\mathbf{r},\mathbf{s}}\rangle$ have $3N$ binary indices and hence the number of such states is exactly equal to the dimension of the Hilbert space; hence they will comprise the full energy spectrum, provided that we can show they are independent. To investigate this question, let us look at the inner product of these states.

Using Eqs. (15) and (14) we find

$$\langle \Psi_{\mathbf{l},\mathbf{r},\mathbf{s}}|\Psi_{\mathbf{l}',\mathbf{r}',\mathbf{s}'}\rangle = \frac{1}{2^N}\langle \Omega_+|B_{\mathbf{l}}\Lambda_{\mathbf{r}',\mathbf{s}'}\Lambda_{\mathbf{r}',\mathbf{s}'}B_{\mathbf{l}'}|\Omega_+\rangle$$
$$= \delta_{\mathbf{l},\mathbf{l}'}\langle \Omega_+|B_{\mathbf{l}}\Lambda_{\mathbf{r}+\mathbf{r}',\mathbf{s}+\mathbf{s}'}|\Omega_+\rangle. \quad (20)$$

The right hand comprises of the inner product of two states, namely, $|\Phi_1\rangle := \Lambda_{\mathbf{r}+\mathbf{r}',\mathbf{s}+\mathbf{s}'}|\Omega_+\rangle$ and $|\Phi_2\rangle := B_{\mathbf{l}}|\Omega_+\rangle$. In view of the structure of $\Lambda_{r,s}$, we see that if $(\mathbf{r}+\mathbf{r}',\mathbf{s}+\mathbf{s}') \neq (0,0)$, then the $|\Phi_1\rangle$ is a linear combination of − spin ups, arranged in open strings, in a sea of spin +'s (from $|\Omega_+\rangle$). For brevity we call this a state of open strings. On the other hand and using the same terminology, the state $|\Phi_2\rangle$ consists of only closed loops. Therefore these two states have no terms in common and their inner products will vanish unless $(\mathbf{r}+\mathbf{r}',\mathbf{s}+\mathbf{s}') = (0,0)$ or equivalently unless $(\mathbf{r},\mathbf{s}) = (\mathbf{r}',\mathbf{s}')$. Moreover by expanding $B_{\mathbf{l}}$ it is clearly seen that $\langle \Omega_+|B_{\mathbf{l}}|\Omega_+\rangle = 1$. Hence we find

$$\langle \Psi_{\mathbf{l},\mathbf{r},\mathbf{s}}|\Psi_{\mathbf{l}',\mathbf{r}',\mathbf{s}'}\rangle = \delta_{\mathbf{l},\mathbf{l}'}\delta_{\mathbf{r},\mathbf{r}'}\delta_{\mathbf{s},\mathbf{s}'}.$$

The independence of these states and the equality of their number with the dimension of Hilbert space indicate that they comprise the complete spectrum of the Hamiltonian. It is also instructive to note the symmetry of the spectrum. If we indicate by $\bar{\mathbf{s}}$ the binary complement of the indices $\mathbf{s}$ (i.e., $\bar{s}_i = 1 + s_i$) and similarly for other indices, we find from Eq. (18) that

$$E_{\mathbf{l},\mathbf{r},\bar{\mathbf{s}}}(J,K) = E_{\mathbf{l},\mathbf{r},\mathbf{s}}(J,K),$$

$$E_{\bar{\mathbf{l}},\bar{\mathbf{r}},\mathbf{s}}(J,K) = E_{\mathbf{l},\mathbf{r},\mathbf{s}}(-J,-K). \quad (21)$$

The first relation expresses the twofold degeneracy, which is the result of the topology of the surface, that is, the action of the cycle W_z. Equation (4) on any state produces another state with the same energy. For this, it is important to note that the new state is independent from the original one, and this is a result of topology. Since the operator W_z flips all the spins around a nontrivial cycle of the manifold (the cylinder), the original state consists of only trivial loops of flipped spins. The other nonlocal operator that commutes with the Hamiltonian, namely, W_x [see Eq. (5)], does not generate another independent state, and it only gives a phase when acting on any eigenstate and hence it does not generate degeneracy.

The second relation indicates how the spectrum is affected if we invert the coupling constants J and K around 0; the spectrum should be inverted around the values $r_j = l_j = 0$. One should also note that only the ground and the top states have twofold degeneracy and the degeneracy of the other states is much larger.

C. Partition function and averages of different observables

From the complete spectrum it is straightforward to calculate the partition function. One writes

$$Z(\beta,J,K) = \mathrm{tr}(e^{-\beta H})$$
$$= \sum_{\mathbf{L},\mathbf{R},\mathbf{S}} \exp -\beta E_{\mathbf{L},\mathbf{R},\mathbf{S}}$$
$$= \sum_{\mathbf{R},\mathbf{S}} \prod_j e^{\beta J R_j (S_{j-1}S_j+1)} \sum_{\mathbf{L}} \prod_j e^{\beta K L_j}$$
$$:= Z_0(\beta,J)Z_1(\beta,K), \quad (22)$$

where the last equality defines the partition functions $Z_0 := Z_0(\beta,J)$ and $Z_1 := Z_1(\beta,K)$. It is obvious that $Z_1 = 2^N \cosh^N \beta K$. Using a transfer matrix, we find

$$Z_0 = \sum_{\mathbf{S}} \prod 2 \cosh \beta J(S_{j-1}S_j + 1)$$
$$= 2^{2N}(\cosh^{2N} \beta J + \sinh^{2N} \beta J). \quad (23)$$

The full partition function is therefore given by

$$Z(\beta,J,K) = 2^{3N}(\cosh^{2N} \beta J + \sinh^{2N} \beta J)\cosh^N \beta K. \quad (24)$$

In the thermodynamic limit, the average energy is obtained from the partition function to be

$$\left\langle \frac{E}{N} \right\rangle = -\frac{\partial}{\partial \beta} \ln Z(\beta,J,K) = -(2J \tanh \beta J + K \tanh \beta K). \quad (25)$$

In the same limit the entropy is found from $S = (1 - \beta \frac{\partial}{\partial \beta}) \ln Z$ to be

$$\frac{S}{N} = 3 \ln 2 + 2 \ln \cosh \beta J + \ln \cosh \beta K - 2\beta J \tanh 2\beta J - \beta K \tanh \beta K. \quad (26)$$

The entropy can be written as a sum of two terms, namely, $S = N[S_s(\beta J) + S_p(\beta K)]$, where the first one is the contribution of the vertex terms and the other is the contribution of plaquette terms, and

$$S_s(x) = 2 \ln 2 + 2 \ln \cosh x - 2x \tanh 2x,$$

$$S_p(x) = \ln 2 + \ln \cosh x - x \tanh x. \quad (27)$$

Figure 2 shows the total entropy as a function of βJ and βK.

An interesting nonlocal observable is the string operator W_x defined in Eq. (4). One finds

$$\langle W_x \rangle = \frac{1}{Z(\beta,J,K)} \sum_{\mathbf{l},\mathbf{r},\mathbf{s}} \langle \Psi_{\mathbf{l}}|\Lambda_{\mathbf{r},\mathbf{s}} W_x \Lambda_{\mathbf{r},\mathbf{s}}|\Psi_{\mathbf{l}}\rangle \exp -\beta E_{\mathbf{l},\mathbf{r},\mathbf{s}}. \quad (28)$$

Passing $\Lambda_{\mathbf{r},\mathbf{s}}$ through W_x, one finds

$$\langle \Psi_{\mathbf{l}}|\Lambda_{\mathbf{r},\mathbf{s}} W_x \Lambda_{\mathbf{r},\mathbf{s}}|\Psi_{\mathbf{l}}\rangle = (-1)^{r_1+r_2+\cdots+r_N}, \quad (29)$$

where we have used the commutation relations of the operators and also the fact that $W_x B_j = B_j W_x$, $\forall j$, and consequently $W_x|\Psi_{\mathbf{l}}\rangle = |\Psi_{\mathbf{l}}\rangle$. Inserting this into Eq. (28) one arrives at

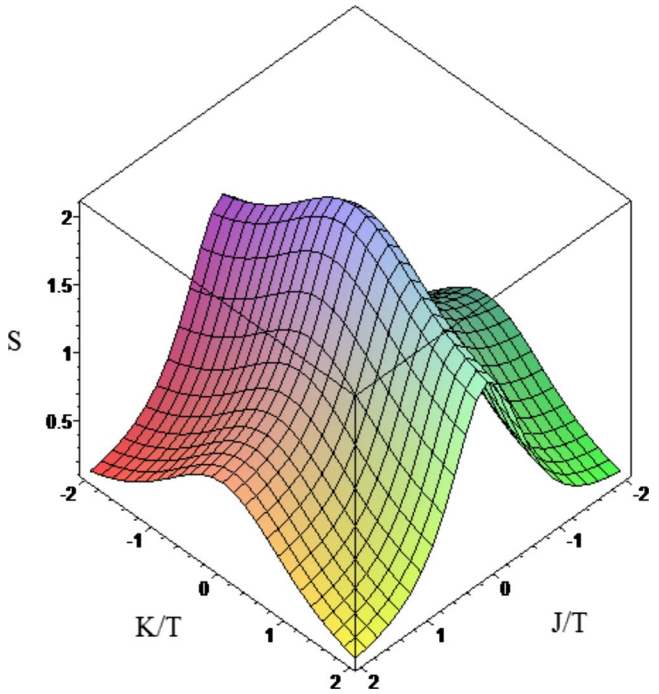

FIG. 2. (Color online) The entropy of the ladder as a function of the couplings and temperature.

$$\langle W_x \rangle = \frac{1}{Z_0} \sum_{\mathbf{R},\mathbf{S}} \prod_j R_j e^{\beta J R_j (S_{j-1} S_j + 1)}, \tag{30}$$

where again an appropriate transfer matrix gives the final result

$$\langle W_x \rangle = \frac{1}{2^{N-1}} \frac{\sinh^N 2\beta J}{\cosh^{2N} \beta J + \sinh^{2N} \beta J}. \tag{31}$$

In the thermodynamic limit this gives

$$\langle W_x \rangle = \begin{cases} 1, & T = 0 \\ 0, & T > 0, \end{cases} \tag{32}$$

which shows a phase transition at zero temperature. It is easy to see that $\langle W_z \rangle = 0$ at all temperatures.

III. REDUCED DENSITY MATRICES AND ENTROPIES OF DIFFERENT SUBSYSTEMS

In this section we derive the reduced density matrices of different subsystems at finite temperature. These subsystems are denoted by A (the spins on one of the legs of the ladder), B (the spins of the totality of all the rungs of the ladder), and C (the spins of a subset of the rungs) (see Fig. 3). The significance of the subsystems B and C is that B corresponds to a topologically nontrivial loop in the surface, while C corresponds to a trivial curve and one expects that this difference of topology shows itself in the entropy of these subsystems. As we will show, this is indeed the case.

To prepare ourselves for the calculation of the reduced density matrix at finite temperatures, in each case we first derive the reduced density matrix of the relevant subsystem

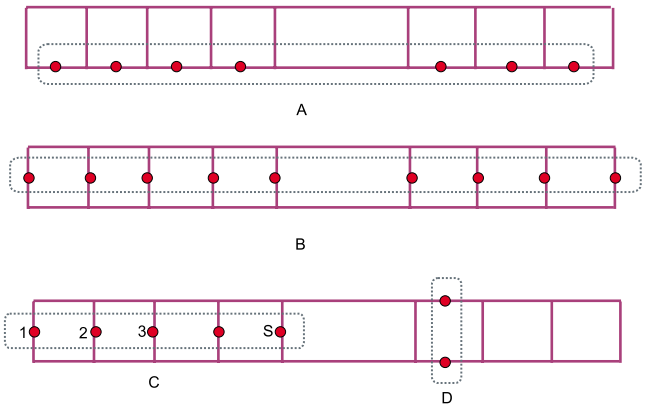

FIG. 3. (Color online) The links with filled circles comprise of the subsystems A-D. At any temperature, the subsystems A, C, and D are in maximally mixed states. Only the subsystem B that has a nontrivial topology has a different state [Eq. (49)].

when the whole system is at the state $|\Psi_{\mathbf{l}}\rangle$. The corresponding density matrices will be denoted by σ_A, σ_B, and σ_C. This will pave the way for determination of the reduced density matrices at arbitrary temperatures, which will be denoted by ρ_A, ρ_B, and ρ_C. We also use the notation $|\Omega_+\rangle_A$ to denote the restriction of the product state $|\Omega_+\rangle$ to the subsystem A with similar notations for B and C.

A. Subsystem A: One leg of the ladder

We have

$$\sigma_A := \mathrm{tr}_{1'',2'',\hat{\ldots},N''}(|\Psi_{\mathbf{l}}\rangle\langle\Psi_{\mathbf{l}}|), \tag{33}$$

where for any subset I, $\hat{I}$ means that we take the trace over the complement of I.

To calculate the trace, we note that the state $|\Psi_{\mathbf{l}}\rangle$ can be written as follows:

$$|\Psi_{\mathbf{l}}\rangle = \frac{1}{\sqrt{2^N}} \sum_{m_1,m_2,\ldots,m_N} (-1)\exp\left(\sum_i l_i m_i\right) B_1^{m_1} \ldots B_N^{m_N} |\Omega_+\rangle, \tag{34}$$

where the indices m_i take the value of 0 or 1. We now use the fact that $B_i = Z_i Z_{i+1} Z_{i'} Z_{i''}$ and take the trace over the product of the B_i operators to arrive at

$$\mathrm{tr}_{1',2',\hat{\ldots},N'}(B_1^{m_1} B_2^{m_2} \ldots B_N^{m_N} |\Omega_+\rangle\langle\Omega_+| B_1^{k_1} B_2^{k_2} \ldots B_N^{k_N}) = Z_{1'}^{m_1} Z_{2'}^{m_2} \ldots Z_{N'}^{m_N} (\langle\Omega_+|M|\Omega_+\rangle)_A Z_{1'}^{k_1} Z_{2'}^{k_2} \ldots Z_{N'}^{k_N}, \tag{35}$$

where

$$M := (Z_1 Z_2 Z_{1''})^{m_1+k_1} (Z_2 Z_3 Z_{2''})^{m_2+k_2} \ldots . \tag{36}$$

From the above equation we find

$$\langle\Omega_+|M|\Omega_+\rangle_A := \delta_{\mathbf{m},\mathbf{k}} = \prod_i \delta_{m_i,k_i}, \tag{37}$$

which when inserted into Eq. (35) this gives

$$\sigma_A = \frac{1}{2^N}\left(\sum_{m_1,\ldots,m_N} Z_{1'}^{m_1}\ldots Z_{N'}^{m_N}|\Omega_+\rangle_A\langle\Omega_+|Z_{1'}^{m_1}\ldots Z_{N'}^{m_N}\right)$$
$$= \left[\frac{1}{2}\left(\sum_m Z^m|+\rangle\langle+|Z^m\right)\right]^{\otimes N} = \frac{1}{2^N}I_A. \tag{38}$$

This shows among other things that at zero temperature, each of the two legs of the ladder is in a maximally mixed state. The interesting point is that this situation persists at all temperatures. To see this, we take the upper leg as our subsystem A since, in this case, the analysis will be greatly simplified. Since the system is symmetric, whatever we obtain will also be valid for the lower leg. We have

$$\mathrm{tr}_{\hat{A}}(|\Psi_{\mathbf{l},\mathbf{r},\mathbf{s}}\rangle\langle\Psi_{\mathbf{l},\mathbf{r},\mathbf{s}}|) = \mathrm{tr}_{\hat{A}}(\Lambda_{\mathbf{r},\mathbf{s}}|\Psi_{\mathbf{l}}\rangle\langle\Psi_{\mathbf{l}}|\Lambda_{\mathbf{r},\mathbf{s}})$$
$$= \mathrm{tr}_{\hat{A}}(|\Psi_{\mathbf{l}}\rangle\langle\Psi_{\mathbf{l}}|) = \frac{1}{2^N}I_A. \tag{39}$$

The reason for taking A to be the upper leg of the ladder is that we could pass through the operator $\Lambda_{\mathbf{r},\mathbf{s}}$ cyclically within the trace and arrive at the simple result that $\rho_A(T)=\frac{1}{2^N}I_A$. Obviously any subsystem of A will also be in a maximally mixed state.

B. Subsystem B: All the rungs of the ladder

Consider now subsystem B as the full set of rungs of the ladder. We first derive the reduced density matrix σ_B when the whole ladder is in the state $|\Psi_{\mathbf{l}}\rangle$. Using decomposition (34) and the structure of B_i operators (2), we find

$$\sigma_B = \frac{1}{2^N}\sum_{m_1,\ldots,m_N,k_1,\ldots,k_N}(-1)\exp\left(\sum_i l_i(m_i+k_i)\right)$$
$$\times_{\hat{B}}\langle\Omega_+|N_{\mathbf{m},\mathbf{k}}|\Omega_+\rangle_{\hat{B}}|\Phi_{\mathbf{m}}\rangle\langle\Phi_{\mathbf{k}}|, \tag{40}$$

where

$$N_{\mathbf{m},\mathbf{k}} := \prod_i (Z_{i'}Z_{i''})^{m_i+k_i} \tag{41}$$

and

$$|\Phi_{\mathbf{m}}\rangle := (Z_1Z_2)^{m_1}\ldots(Z_NZ_1)^{m_N}|\Omega_+\rangle_B. \tag{42}$$

Using the fact that $_{\hat{B}}\langle\Omega_+|N_{\mathbf{m},\mathbf{k}}|\Omega_+\rangle_{\hat{B}}=\delta_{\mathbf{m},\mathbf{k}}$, inserting the result in Eq. (40), noting the two-to-one correspondence between the indices m_i and the powers of Z_i, and rearranging terms, we obtain

$$\sigma_B = \frac{1}{2^{N-1}}\sum_{q_1,\ldots,q_{N-1}}|\tilde{\Phi}_{\mathbf{q}}\rangle\langle\tilde{\Phi}_{\mathbf{q}}|, \tag{43}$$

where

$$|\tilde{\Phi}_{\mathbf{q}}\rangle := Z_1^{q_1}Z_2^{q_2}\ldots Z_N^{q_1+q_2+\cdots+q_{N-1}}|\Omega_+\rangle_B. \tag{44}$$

Note that this is independent of the index set $\mathbf{l}$ of the state $|\Psi_{\mathbf{l}}\rangle$. Also in each state $|\tilde{\Phi}_{\mathbf{q}}\rangle$ the flip operators come in pair, so this state is an even parity state, i.e., a state where an even number of spins has been flipped from + to −. The state σ_B is thus a uniform mixture of all even parity states. We call this density matrix σ^{even}. Let us now consider finite temperatures, for which we have to calculate

$$\mathrm{tr}_{1,2,\hat{\ldots},N}(|\Psi_{\mathbf{l},\mathbf{r},\mathbf{s}}\rangle\langle\Psi_{\mathbf{l},\mathbf{r},\mathbf{s}}|) = \mathrm{tr}_{1,2,\hat{\ldots},N}(\Lambda_{\mathbf{r},\mathbf{s}}|\Psi_{\mathbf{l}}\rangle\langle\Psi_{\mathbf{l}}|\Lambda_{\mathbf{r},\mathbf{s}})$$
$$= \Lambda_{\mathbf{r}}\sigma^{(even)}\Lambda_{\mathbf{r}}, \tag{45}$$

where $\Lambda_{\mathbf{r}}=\Pi_i Z_i^{r_i}$. The reduced density matrix at finite temperature will now be given by

$$\rho_B(T) = \frac{1}{Z_0}\sum_{\mathbf{r},\mathbf{s}}\exp-\beta E_{\mathbf{r},\mathbf{s}}\Lambda_{\mathbf{r}}\sigma^{even}\Lambda_{\mathbf{r}}. \tag{46}$$

From the above definition of $\Lambda_{\mathbf{r}}$ and that of σ^{even} and $|\tilde{\Phi}_{\mathbf{q}}\rangle$, one finds that

$$\Lambda_{\mathbf{r}}\sigma^{(even)}\Lambda_{\mathbf{r}} = \begin{cases}\sigma^{even}, & |\mathbf{r}|=0\\ \sigma^{odd}, & |\mathbf{r}|=1,\end{cases} \tag{47}$$

where σ^{odd} is the uniform mixture of odd-parity states and $|\mathbf{r}|$ denotes the degree of $\mathbf{r}$, i.e., $|\mathbf{r}|=r_1+r_2+\cdots+r_N$.

Inserting this into Eq. (46) yields

$$\rho_B(T) = \frac{1}{2Z_0}\sum_{\mathbf{r},\mathbf{s}}\exp-\beta E_{\mathbf{r},\mathbf{s}}\{(1+(-1)^{|\mathbf{r}|})\sigma^{even}$$
$$+[1-(-1)^{|\mathbf{r}|}]\sigma^{odd}\}. \tag{48}$$

Using the expression of the parity $|\mathbf{r}|$ and Eqs. (28) and (29) we find the following simple expression:

$$\rho_B(T) = \frac{1}{2}(1+\langle W_x\rangle)\sigma^{even}+\frac{1}{2}(1-\langle W_x\rangle)\sigma^{odd}$$
$$= \frac{1}{2^N}I_B+\frac{1}{2}\langle W_x\rangle(\sigma^{even}-\sigma^{odd}), \tag{49}$$

where use has been made of the fact that $\frac{1}{2}(\sigma^{even}+\sigma^{odd})=\frac{1}{2^N}I_B$. The entropy of this state, which is a mixture of orthogonal states, can now be readily calculated. A straightforward calculation gives

$$S(\rho_B) = N-1+H\left(\frac{1+\langle W_x\rangle}{2}\right), \tag{50}$$

where $H(p)=-p\log_2 p-(1-p)\log_2(1-p)$ is the Shannon entropy function.

Figure 4 shows $S_{\rho_B}-(N-1)$ for several values of system sizes N. In the thermodynamic limit, there is a sharp rise in this quantity only at zero temperature, but for finite N, it is also seen that there is an almost sharp rise at finite temperatures.

C. Subsystem C: A subset of the rungs

Once the density matrix of the subsystem B is obtained, we can trace out any number of the rungs to find the reduced

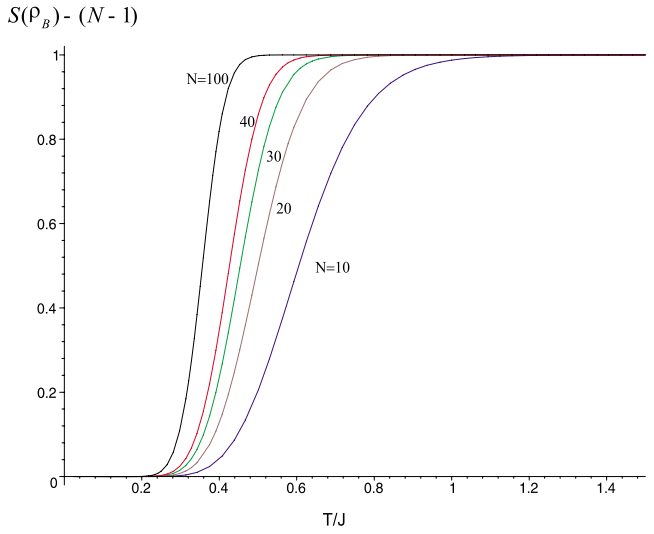

FIG. 4. (Color online) Thermal entanglement of subsystem B with the rest of the lattice as a function of temperature for different system sizes. The system sizes are 10, 20, 30, 40, and 100.

density matrix of the remaining subset of rungs. Using Eq. (49) and noting that on taking the trace over any subsystem the contributions of σ^{even} and σ^{odd} cancel each other, one arrives at the simple result that

$$\rho_C = \frac{1}{2^{|C|}} I_C, \quad \forall\, T, \tag{51}$$

where C is any proper subset of the rungs and $|C|$ is the size of this subset. It is interesting to note that how the topology of the surface is reflected in the entropy of its subsystems.

D. Thermal entanglement of two spins

From the reduced density matrices found in previous subsections, we know that any two spins on a leg of the ladder are in a maximally mixed state $\rho=\frac{1}{4}I$ and hence there is no thermal entanglement between such spins. The same is true between any two spins on the rungs of the ladder. In fact, it has been shown[28] that in the ground state, there is no entanglement between any two qubits. However one can ask if at higher temperature some degree of entanglement is caused by thermal fluctuations. This indeed happens in some spin systems below a certain threshold temperature. To investigate this, we compute the reduced density matrices of the two spins, say, $1'$ and $1''$ in Fig. 3 on the two legs, opposite to each other. We call this subsystem D. The first step is to calculate $\mathrm{tr}_{\hat{D}}|\psi_{\mathbf{l},\mathbf{r},\mathbf{s}}\rangle\langle\psi_{\mathbf{l},\mathbf{r},\mathbf{s}}|$ that is equal to

$$\frac{1}{2^N}\mathrm{tr}_{\hat{D}}(\Lambda_{\mathbf{r},\mathbf{s}}B_{\mathbf{l}}|\Omega_+\rangle\langle\Omega_+|\Lambda_{\mathbf{r},\mathbf{s}}B_{\mathbf{l}}) = \frac{1}{2^N}Z_1^{s_1}\,\mathrm{tr}_{\hat{D}}(B_{\mathbf{l}}|\Omega_+\rangle\langle\Omega_+|B_{\mathbf{l}})Z_{1'}^{s_1}.$$

In calculating the trace, one can use the cyclic property of the trace and move around all the terms $[1+(-1)^{l_i}B_i]$ except the term $1+(-1)^{l_1}B_1$ (which acts nontrivially on the space D) and use the property $[1+(-1)^{l_i}B_i]^2=2[1+(-1)^{l_i}B_i]$, which after some algebra gives

$$\begin{aligned}
&\mathrm{tr}_{\hat{D}}(B_{\mathbf{l}}|\Omega_+\rangle\langle\Omega_+|B_{\mathbf{l}}) \\
&= 2^{N-1}\,\mathrm{tr}_{\hat{D}}\Big\{\prod_{i\neq 1}[1+(-1)^{l_i}B_i][1+(-1)^{l_1}B_1] \\
&\quad\times|\Omega_+\rangle\langle\Omega_+|[1+(-1)^{l_1}B_1]\Big\} \\
&= 2^{N-1}\,\mathrm{tr}_{\hat{D}}\{[1+(-1)^{l_1}B_1]|\Omega_+\rangle\langle\Omega_+|[1+(-1)^{l_1}B_1]\} \\
&= 2^{N-1}(|++\rangle\langle++|+|--\rangle\langle--|)_{1',1''},
\end{aligned} \tag{52}$$

where in the second line we have used the fact that the closed loops generated by any of B_i's cannot be compensated by B_1 to make a nonvanishing trace and the third line is the result of explicit expansion and calculation. We will then have

$$\rho_D = \frac{1}{2Z}\sum_{\mathbf{l},\mathbf{r},\mathbf{s}} \exp-\beta E_{\mathbf{l},\mathbf{r},\mathbf{s}} Z_1^{s_1}(|++\rangle\langle++|+|--\rangle\langle--|)_{1',1''}Z_{1'}^{s_1}. \tag{53}$$

Acting the operators $Z_1^{s_1}$ on both sides and performing the above simple calculation with the help of the transfer matrix, we find that $\rho_D=\frac{1}{4}I_D$, which means that there is no thermal entanglement between these two spins.

IV. COMPLETE SPECTRUM OF THE THREE-LEG LADDER

What has been done for the two-leg ladder can be extended to three-leg ladder (Fig. 5) without much effort. The first step is to define the states

$$|\Psi_{\mathbf{l}}\rangle = \frac{1}{2^N}\prod_p [1+(-1)^{l_p}B_p]|\Omega_+\rangle, \tag{54}$$

and then the trick is to find a suitable generalization for the operators $\Lambda_{\mathbf{r},\mathbf{s}}$, so that their action on the above state produces the correct number of independent eigenstates of the vertex operators. The suitable generalization is as follows:

$$\Lambda_{\mathbf{r},\mathbf{s},\mathbf{t}} := \prod_{i=1}^{N} Z_i^{r_i} Z_{i'}^{s_i} Z_{i''}^{t_i}, \tag{55}$$

where the flipping operators correspond to the links shown in Fig. 6.

One can now easily verify the following commutation relations, where we use A_j^+, A_j^0, and A_j^- for vertex operators on site j for the lower, middle, and upper legs of the ladder, respectively,

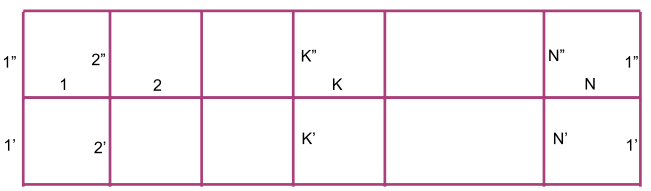

FIG. 5. (Color online) The labeling used in the text for the links on the three-leg ladder.

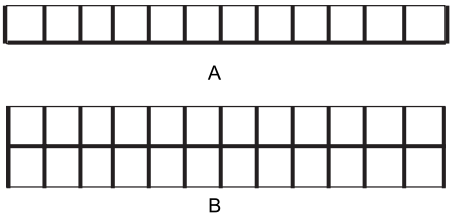

FIG. 6. The flipping operators are chosen form the set of links shown in bold, (A) for the two-leg ladder and (B) for the three-leg ladder.

$$A_j^+ \Lambda_{\mathbf{r,s,t}} = (-1)^{r_j} \Lambda_{\mathbf{r,s,t}} A_j^+,$$

$$A_j^0 \Lambda_{\mathbf{r,s,t}} = (-1)^{s_{j-1}+s_j+r_j+t_j} \Lambda_{\mathbf{r,s,t}} A_j^0,$$

$$A_j^- \Lambda_{\mathbf{r,s,t}} = (-1)^{t_j} \Lambda_{\mathbf{r,s,t}} A_j^-. \tag{56}$$

One can proceed along the same way as detailed in Sec. II for the two-leg ladder and show that the states $|\Psi_{\mathbf{l,r,s,t}}\rangle := \Lambda_{\mathbf{r,s,t}}|\Psi_{\mathbf{l}}\rangle$ are energy eigenstates with energies given by

$$E = -J\sum_i (R_i T_i S_{i-1} S_i + R_i + T_i) - K\sum_i L_i, \tag{57}$$

where again we have used Ising-like variables, i.e., $T_i := (-1)^{t_i}$ instead of the binary variables. Moreover the number of these states is 2^{5N} that is equal to the dimension of the Hilbert space and they are orthogonal [see the reasoning following Eq. (20)].

The basic point is that every conceivable combinations of the flipping operators in $\Lambda_{\mathbf{r,s,t}}$, corresponding to Fig. 6 when acting on $|\Omega_+\rangle$, will produce only open or homologically trivial loops of negative spins that will certainly be orthogonal to the state generated by $B_{\mathbf{l}}$ acting on $|\Omega_+\rangle$. The partition function turns out to be

$$Z(\beta,J,K) = 2^{5N} \cosh^{2N} \beta K [\cosh^{3N} \beta J + \sinh^{3N} \beta J]. \tag{58}$$

V. GENERALIZATION TO TWO-DIMENSIONAL SQUARE LATTICE

To what extent this study can be pursued for the two-dimensional lattice (i.e., a torus)? On a lattice with N^2 sites and $2N^2$ links, the Hilbert space dimension is 2^{2N^2}. We can already construct 2^{N^2} (un-normalized) states of the form $|\Psi_{\mathbf{l}}\rangle := \Pi_p[1+(-1)^{l_p} B_p]|\Omega_+\rangle$, which are energy eigenstates. To find more states, we have to find subsets I of flipping operators and then construct operators of the form $\Lambda_{\mathbf{s}} := \Pi_{i\in I} Z_i^{s_i}$ and energy eigenstates as $|\Psi_{\mathbf{s,l}}\rangle = \Lambda_{\mathbf{s}} B_{\mathbf{l}} |\Psi_{\mathbf{l}}\rangle$. The subset I should have the following important property: when acting on $|\Omega_+\rangle$, no combination of links in I should be able to generate a homologically trivial loop of negative spins on the lattice. Let I_{max} be a maximal set of this type with $|I_{max}|$ elements. Then the total number of independent energy eigenstates found in this way is $2^{N^2+|I_{max}|}$.

For the square lattice of N^2 sites, one such set is shown in Fig. 7, where $|I_{max}| = N^2$. In fact, I_{max} is nothing but a one

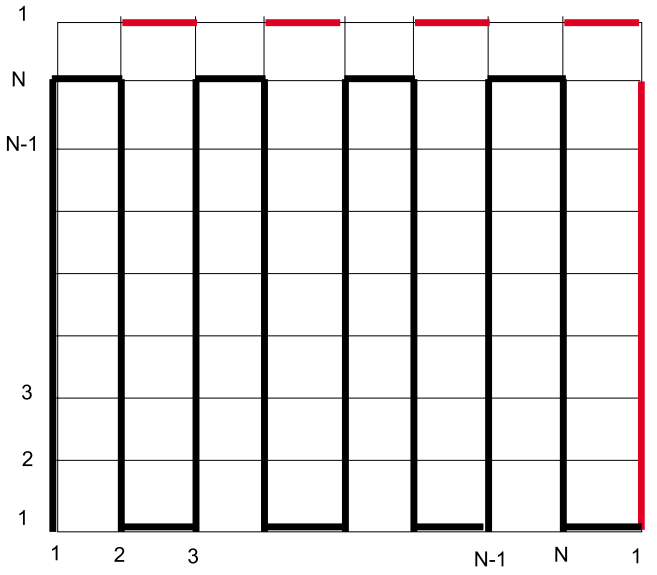

FIG. 7. (Color online) The excitation curve for the two-dimensional lattice, the curve from which the flipping operators are chosen for creation of the energy eigenstates. The links in red (the ones in the top and right of the lattice) indicate identical links on the other sides of the lattice due to the torus topology.

cycle that goes back and forth around the torus but does not wrap around it and comprises half of the links on the network. We call the maximal cycle since it is the cycle that contains the maximal set of links (the addition of any link to this cycle will make a trivial loop out of it) or the excitation curve since flipping operators chosen from it and acting on $|\Psi_{\mathbf{l}}\rangle$ create all the excited states.

Since $|I_{max}| = N^2$, the states constructed as $|\Psi_{\mathbf{m,l}}\rangle = \Lambda_{\mathbf{m}} B_{\mathbf{l}} |\Psi_{\mathbf{l}}\rangle$ form the whole set of energy eigenstates. Numbering the links along the boldface curve shown in Fig. 7, in a consecutive way from 1 to N^2, shows that the energy of such a state is equal to $E_{\mathbf{s,l}} = -J\Sigma_{i=1}^{N^2} S_i S_{i+1} - K\Sigma_i L_i$, where we have used the Ising-type labels $S_i := (-1)^{s_i}$ and $L_i = (-1)^{l_i}$ instead of the binary labels s_i and l_i. This leads to the partition function

$$Z_{\text{square lattice}}(\beta,J,K) = 2^{2N^2} \cosh^{N^2} \beta K [\cosh^{N^2} \beta J + \sinh^{N^2} \beta J]. \tag{59}$$

Remark 1. We could have taken just such a canonical curve for the two and three leg ladders instead of the ones shown in Fig. 6, although it may have rendered the calculations of reduced density matrices in these simple cases unnecessarily involved. The curves shown in Fig. 6 can be obtained from this canonical curve by the moving up and down the horizontal links appropriately, which amounts to application of plaquette operators. Such plaquette operators affect the eigenstate by only a phase.

Remark 2. As is clear from Eq. (58) and the argument preceding it, the partition function of the Kitaev model is the same as that of two noninteracting chains of spins in magnetic fields of strength J and K. This is particularly clear from Eq. (58) if we take the thermodynamic limit. Note that there is not even a nearest-neighbor interaction between the

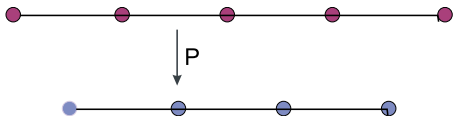

FIG. 8. (Color online) The projection operator P that maps the Hilbert space of two neighboring sites onto the Hilbert space of one link turns the Ising Hamiltonian into a chain of noninteracting spins in an external magnetic field.

spins in these chains and only magnetic fields J and K control the interactions. Does this imply that a local redefinition of degrees of freedom can map the Kitaev model to such a simple system? Although its possibility cannot be ruled out, it is far from clear that such a mapping, even if it exists, has a simple form. Let us look at the very definition of Hamiltonian (1). The partition function turns out to be

$$Z = \mathrm{tr}\left\{\exp\left[\beta\left(J\sum_s A_s + K\sum_p B_p\right)\right]\right\} = \mathrm{tr}\left(\prod_s e^{\beta J A_s}\prod_p e^{\beta K B_p}\right), \tag{60}$$

where the last equality is a result of the commutativity of all the operators A_s and B_p that act on the links of the lattice. If we could find an isomorphism between the 2^{2N^2} dimensional Hilbert space of the lattice and another one where the operators A_s and B_p could be represented as tensor product operators with disjoint support, i.e., $A_s := I^{\otimes s-1}\otimes A\otimes I^{\otimes(N^2-s)}$, then the above trace could be broken to $\Pi_s \mathrm{tr}(e^{\beta J A_s})\Pi_p \mathrm{tr}(e^{\beta K B_p})$ directly leading to Eq. (59) in the thermodynamic limit.

To gain further insight into this problem, let us look at the Ising model in a one-dimensional open chain, defined by the Hamiltonian

$$H_{Ising} = J\sum_{k=1}^{N-1}\sigma_{z,k}\sigma_{z,k+1}. \tag{61}$$

The Hilbert space consists of tensor product of two-dimensional spaces on the vertices of the chain, with basis states $|+\rangle$ and $|-\rangle$. Inspired by the form of the interaction energy that gives equal energies to the states $|++\rangle$ and $|--\rangle$ or to the states $|+-\rangle$ and $|-+\rangle$, one can construct a new Hilbert space comprised of tensor product of states on the links (Fig. 8). Such a mapping is done by local projection operators of the form

$$P := |1\rangle(\langle++|+\langle--|) + |-1\rangle(\langle+-|+\langle-+|). \tag{62}$$

These projection operators turn the original Hilbert space on the vertices to the new Hilbert space on the links and the bilocal operators $\sigma_{z,k}\sigma_{z,k+1}$ on the neighboring vertices to local operators $\sigma_{z,i}$ on the links. Ising Hamiltonian (61) will be projected to that of a simple chain of noninteracting spins in an external magnetic field J.

It is obvious that such a mapping is not possible for the one-dimensional periodic chain or for the two-dimensional Ising model. In view of the simple form of the partition function of the Kitaev model and its spectrum, defined in this section, it may be possible to find such a map (homomorphisms between the Hilbert spaces), however, I guess that this is a highly nontrivial task. In this regard, Fig. 7 may give a clue as to how such a mapping should be defined.

VI. DISCUSSION

We have determined the complete spectrum of the Kitaev model on a spin ladder and from there we have determined the reduced density matrices for its various subsystems at finite temperature. We have shown that on two- and three-leg ladders, the model is equivalent to particular types of one-dimensional classical Ising models, models with different spins on the sites and links. The calculations in these models have led to the insight for obtaining the spectrum of the two-dimensional lattice.

Knowing the full spectrum in this way will enable us to study the entanglement and many other properties of the Kitaev model in detail. Furthermore this knowledge may be useful in other more detailed studies of the Kitaev model, where the dynamics of the model is required, for example, in the study of dynamics of classical and quantum phase transitions in systems with topological order, mentioned in Sec. I.[30] Another interesting context is the study of autocorrelation times of the toric code, which is related to the important problem of how long quantum information can be protected in topological degrees of freedom in a background of inevitable thermal fluctuations. This later problem was first addressed in Refs. 32 and 33, in which among other things a mapping of the spectrum of the Kitaev model to two uncoupled Ising chains was found.

Another important problem that may be treated in an alternative way by the characterization of spectrum in the way shown in this paper is the problem of sustainment of topological order at finite temperatures. This problem has been studied in a number of works[24,34,35] using a different description of the spectrum.

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